ON THE NATURE OF MATHEMATICAL KNOWLEDGE.

"MATHEMATICALLY certain and indubitable" is a phrase which is often heard in the sciences and in common life, to express the idea that the seal of truth is more deeply imprinted upon a proposition than is the case with ordinary acts of knowledge. We propose to investigate here to what extent mathematical knowledge really is more certain and unequivocal than other knowledge.

The intrinsic character of mathematical research and knowledge is based essentially on three properties: first, on its conservative attitude towards the old truths and discoveries of mathematics; secondly, on its progressive mode of development, due to the incessant acquisition of new knowledge on the basis of old knowledge; and thirdly, on its self-sufficiency and its consequent absolute independence.

That mathematics is the most conservative of all the sciences is apparent from the incontestability of its propositions. This last character bestows on mathematics the enviable superiority that no new development can undo the work of previous developments or substitute new in the place of old results. The discoveries that Pythagoras, Archimedes, and Apollonius made are as valid to-day as they were two thousand years ago. This is a trait which no other science possesses. The notions of previous centuries regarding the nature of heat have been disproved. Goethe's theory of colors is now antiquated. The theory of the binary combination of salts was supplanted by the theory of substitution, and this, in its turn, has also given way to newer conceptions. Think of the pro-

found changes which the conceptions of theoretical medicine, zoölogy, botany, mineralogy, and geology have undergone. It is the same, too, in the other sciences. In philology, comparative linguistics, and history our ideas are quite different from what they formerly were.

In no other science is it so indispensable a condition that whatever is asserted must be true, as it is in mathematics. Whenever, therefore, a controversy arises in mathematics, the issue is not whether a thing is true or not, but whether the proof might not be conducted more simply in some other way, or whether the proposition demonstrated is sufficiently important for the advancement of the science as to deserve especial enunciation and emphasis, or finally, whether the proposition is not a special case of some other and more general truth which is just as easily discovered.

Let me recall the controversy which has been waged in this century regarding the eleventh axiom of Euclid, that only one line can be drawn through a point parallel to another straight line. This discussion impugned in no wise the truth of the proposition; for that things are true in mathematics is so much a matter of course that on this point it is impossible for a controversy to arise. The discussion merely touched the question whether the axiom was capable of demonstration solely by means of the other propositions, or whether it was not a special property, apprehensible only by sense-experience, of that space of three dimensions in which the organic world has been produced and which therefore is of all others alone within the reach of our powers of representation. The truth of the last supposition affects in no respect the correctness of the axiom but simply assigns to it, in an epistemological regard, a different status from what it would have if it were demonstrable, as was one time thought, without the aid of the senses, and solely by the other propositions of mathematics.

I may recall also a second controversy which arose a few decades ago as to whether all continuous functions were differentiable. In the outcome, continuous functions were defined that possessed no differential coefficient, and it was thus learned that certain truths which were enunciated unconditionally by Newton, Leibnitz, and

their mathematical successors, required qualification. But this did not invalidate at all the correctness of the method of differentiation and its application in all practical cases; the theoretical speculations pursued on this subject simply clarified ideas and sifted out the conditions upon which differentiability depended. Happily the gifted minds who create the new methods and open up the new paths of research in mathematics, are not deterred by the fear that a subsequent generation gifted with unusual acumen will spy out isolated cases in which their methods fail. Happily the creators of the differential calculus pushed onward without a thought that a critical posterity would discover exceptions to their results. In every great advance that mathematics makes, the clarification and scrutinisation of the results reached are reserved necessarily for a subsequent time, but with it the demonstration of those results is more rigorously established. Despite all this, however, in no science does cognition bear so unmistakably the imprint of truth as in pure mathematics. And this fact bestows on mathematics its conservative character.

This conservative character again is displayed in the *objects* of mathematical research. The physician, the historian, the geographer, and the philologist have to-day quite different fields of investigation from what they had centuries ago. In mathematics, too, every new age gives birth to new problems, arising partly from the advance of the science itself, and partly also from the advance of civilisation, where improvements in the other sciences bring in their train the formulation of problems that are constantly taxing anew the resources of mathematics. But despite all this, in mathematics more than in any other science problems exist that have played a rôle for hundreds, nay, for thousands of years.

In the oldest mathematical manuscript which we possess, the Papyrus Rhind of the British Museum, which dates back to the eighteenth century before Christ, and whose decipherment we owe to the industry of Eisenlohr, we find an attempt to solve the problem of converting a circle into a square of equal area, a problem whose history covers a period of three and a half thousand years. For it was not until 1882 that a rigorous proof was given of the im-

possibility of solving this problem exactly by the use of compasses and ruler alone.

It is, of course, the insoluble problems that have the longest history; partly because it is harder to show that a thing is impossible than that it is possible, and, on the other hand, because problems that have long defied solution are ever evoking anew the spirit of inquiry and the ambition of mathematicians, and because the uncertainty of insolubility lends to such problems a peculiar charm. Of the geometrical problems that have occupied competent and incompetent minds from the time of the ancient Greeks to the present may be mentioned in addition to the squaring of the circle two others that are also perhaps well-known to educated readers, at least by name: the trisection of the angle and the Delic problem of the duplication of the cube. All three problems involve the condition, which is often overlooked by unprofessional readers, that only compasses and ruler shall be employed in the constructions. In the trisection of the angle any angle is assigned, and it is required to find the two straight lines which divide the angle into three equal parts. In the Delic problem the edge of a cube is given and the edge of a second cube is sought, containing twice the volume of the first cube. In Greece, in the golden age of the sciences, when all scholars had to understand mathematics, it was a fashionable requisite almost to have employed oneself on these famous problems.

Fortunately for us, these problems were insoluble. For in their ambition to conquer them it came to pass that men busied themselves more and more with geometry, and in this way kept constantly discovering new truths and developing new theories, all of which perhaps might never have been done if the problems had been soluble and had early received their solutions. Thus is the struggle after truth often more fruitful than the actual discovery of truth. So, too, although in a slightly different sense, the apophthegm of Lessing is confirmed here, that the search for truth is to be preferred to its possession.

Whilst the three above-named problems are now acknowledged to be insoluble, and have ceased, therefore, to stimulate mathematical inquiry, there are of course other problems in mathematics

whose solution has been sought for a long time, but not yet reached, and in the case of which there is no reason for supposing that they are insoluble. Of such problems the two following perhaps have found their way out of the isolated circles of mathematicians and have become more or less known to other scholars. I refer to the astronomical Problem of Three Bodies and to the problem of the frequency of prime numbers. The first of these two problems assumes three or more heavenly bodies whose movements are mutually influenced by one another according to Newton's law of gravitation, and requires the exact determination of the path which each body describes. The second problem requires the construction of a formula which shall tell how many prime numbers there are below a certain given number. So far these two problems have been solved only approximately, and not with absolute mathematical exactness.

If the eternal and inviolable correctness of its truths lends to mathematical research, and therefore also to mathematical knowledge, a *conservative* character, on the other hand, by the continuous outgrowth of new truths and methods from the old, *progressiveness* is also one of its characteristics. In marvellous profusion old knowledge is augmented by new, which has the old as its necessary condition, and, therefore, could not have arisen had not the old preceded it. The indestructibility of the edifice of mathematics renders it possible that the work can be carried to ever loftier and loftier heights without fear that the highest stories shall be less solid and safe than the foundations, which are the axioms, or the lower stories, which are the elementary propositions. But it is necessary for this that all the stones should be *properly fitted together*; and it would be idle labor to attempt to lay a stone that belonged above in a place below. A good example of a stone of this character belonging in what is now the uppermost layer of the edifice, is Lindemann's demonstration of the insolubility of the quadrature of the circle, a demonstration of which interesting simplifications have been given by several mathematicians, including Weierstrass and Felix Klein. Lindemann's demonstration could not have been produced in the preceding century, because it rests necessarily on theories whose development falls in the present century. It is true,

Lambert succeeded in 1761 in demonstrating the irrationality of the ratio of the circumference of a circle to its diameter, or, which is the same thing, the irrationality of the ratio of the area of a circle to the area of the square on its radius. Afterwards, Lambert also supplied a proof that it was impossible for this ratio to be the square root of a rational number. But this was the first step only in a long journey. The attempt to prove that the old problem is insoluble was still destined to fail. An astounding mass of mathematical investigations were necessary before the demonstration could be successfully accomplished.

As we see, the majority of the mathematical truths now possessed by us presuppose the intellectual toil of many centuries. A mathematician, therefore, who wishes to-day to acquire a thorough understanding of modern research in this department, must think over again in quickened tempo the mathematical labor of several centuries. This constant dependence of new results on old ones stamps mathematics as a science of uncommon exclusiveness and renders it mostly impossible to open up to uninitiated readers a speedy path to the apprehension of the higher mathematical truths. For this reason, too, the theories and results of mathematics are rarely adapted for popular presentation. There is no royal road to the knowledge of mathematics, as Euclid once said to the first Egyptian Ptolemy. This same inaccessibility of mathematics, although it secures for it a lofty and aristocratic place among the sciences, also renders it odious to those who have never learned it, and who dread the great labor involved in acquiring an understanding of the questions of modern mathematics. Neither in the languages nor in the natural sciences are the investigations and results so closely interdependent as to make it impossible to acquaint the uninitiated student with single branches or with particular results of these sciences, without causing him to go through a long course of preliminary study.

The third trait which distinguishes mathematical research is its self-sufficiency. In philology the field of inquiry is the organic one of languages, and philology, therefore, is dependent in its investigations on the mode of development of languages, which is more or

less accidental. Its task is connected with something which is given to it from without and which it cannot alter. It is much the same with the science of history, which must contemplate the history of mankind as it has actually occurred. Also zoölogy, botany, mineralogy, geology, and chemistry work with given data. In order not to become involved in futile speculations the last-mentioned sciences are constantly and inevitably obliged to revert to observations by the senses. It is then their task to link together these individual observations by bonds of causality and in this way to erect from the single stones an edifice, the view of which will render it easier for limited human intelligence to comprehend nature. Physics of all sciences stands nearest to mathematics in this respect, because unlike the other sciences she is generally in need of only a few observations in order to proceed deductively. But physics, too, must resort to observations of nature, and could not do without them for any length of time.

Mathematics alone, after certain premises have been permanently established, is able to pursue its course of development independently and unmindful of things outside of it. It can leave entirely unnoticed, questions and influences emanating from the outer world, and continue nevertheless its development.

As regards geometry, the first beginnings of this science did indeed take their origin in the requirements of practical life. But it was not long before it freed itself from the restrictions of the practical art to which it owed its birth. Herodotus recounts that geometry had its origin in Egypt where the inundations of the Nile obliterated the boundaries of the riparian estates, and by giving rise to frequent disputes constantly compelled the inhabitants to compare the areas of fields of different shapes. But with the early Greek mathematicians, who were the heirs of the Egyptian art of measurement, geometry appeared as a science which men pursued for its own sake without a thought of how their intellectual discoveries could be turned to practical account.

Nevertheless, although the workers in the domain of pure mathematics are not stimulated by the thought that their researches are likely to be of practical value, yet that result is still frequently real-

ised, often after the lapse of centuries. The history of mathematics shows numerous instances of mathematical results which were originally the outcome of a mere desire to extend the science, suddenly receiving in astronomy, mechanics, or in physics practical applications which their originators could never have dreamt of. Thus Apollonius erected in ancient times the stately edifice of the properties of conic sections, without having any idea that the planets moved about the sun in conic sections, and that a Kepler and a Newton were one day to come who should apply these properties to explaining and calculating the motions of the planets about the sun. The question of the practical availability of its results in other fields has at no period exercised more than a subordinate influence on mathematical inquiry. Particularly is this true of *modern* mathematical research, whether the same consist in the extended development of isolated theories or in the uniting under a higher point of view of theories heretofore regarded as different.[1]

This independence of its character has rendered the results of pure mathematics independent also of the accidental direction which the development of civilisation has taken on our planet; so that the remark is not altogether without justification, that if beings endowed with intelligence existed on other planets, the truths of mathematics would afford the only basis of an understanding with them. Uninterruptedly and wholly from its own resources mathematics has built itself up. It is scarcely credible to a person not versed in the science, that mathematicians can derive satisfaction from the comfortless and wearisome operation of heaping up demonstration on demonstration, of rivetting truth on truth, and of tormenting themselves with self-imposed problems, whose solution stands no one in stead, and affords satisfaction to no one but the solver himself. Yet this self-sufficiency of mathematicians becomes a little more intelligible when we reflect that the progress which has been made, particularly in the last few decades, and which is uninfluenced from without, does not consist solely in the accumulation of new truths

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[1] Cf. Felix Klein, "Remarks Given at the Opening of the Mathematical and Astronomical Congress at Chicago." *The Monist* (Vol. IV, No. 1, October, 1893).

and in the enunciation of new problems, nor solely in deductions and solutions, but culminates rather in the discovery of new methods and points of view in which the old disconnected and isolated results appear suddenly in a new connexion or as different interpretations of a common fundamental truth, or finally, as a single organic whole.

Thus, for example, the idea of representing imaginary and complex numbers in a plane, two apparently different branches, the theory of dividing the circumference of a circle into any given number of equal parts, and the theory of the solutions of the equation $x^n = 1$, have been made to exhibit an extremely simple connexion with one another which enables us to express many a truth of algebra in a corresponding truth of geometry and *vice versa*. Another example is afforded by the discovery which we chiefly owe to Alfred Clebsch, of the relation which subsists between the higher theory of functions and the theory of algebraic curves, a relation which led to the discovery of the condition under which two curves can be co-ordinated to each other, point for point, and hence also adequately represented on each other. Of course such combinations and extensions of view possess a much greater charm for the mathematician than the mere accumulation of truths and solutions, whose fascination consists entirely in their truth or correctness.

From these three cardinal characteristics, now, which distinguish mathematical *research* from research in other fields, we may gather at once the three positive characteristics that distinguish mathematical *knowledge* from other knowledge. They may be briefly expressed as follows; first, mathematical knowledge bears more distinctly the imprint of truth on all its results than any other kind of knowledge; secondly, it is always a sure preliminary step to the attainment of other correct knowledge; thirdly, it has no need of other knowledge. Naturally, however, there are associated with these characteristics which place mathematical knowledge high above all other knowledge, other characteristics which somewhat counterbalance the great superiority which mathematics thus appears to have over the other sciences. In order to show more distinctly the nature of these characteristics, which we prefer to call

negative, we shall select and confine our remarks to a branch which is commonly taken to be synonymous with mathematics, namely, to arithmetic in the broadest sense of the word.

The subject of inquiry in arithmetic is numbers and their combinations. On this account arithmetic is, of all sciences, most free from what lies outside its boundaries. Perception by the senses is necessary only in an extremely insignificant measure for the understanding of its definitions and premises. It is possible to acquaint a person who lacks both sight and hearing with the fundamental principles of arithmetic solely by the medium of "time." Such a person needs only the sense of feeling. By slight excitations of his skin, induced at equal or unequal intervals of time, he can be led to the notion of differences of time and hence also to the notion of differences of number. Uninfluenced by matter and force, independently, too, of the properties of geometrical magnitudes, arithmetic could be conducted solely by its own intrinsic potencies to its highest goals, drawing deductively truth from truth, without a break.

But what sort of a science should we arrive at by this method of procedure? Nothing but a gigantic web of self-evident truths. For, once we admit the first notions and premises to which a man thus bereft of his senses can be led, we are compelled of necessity also to admit the derivative results of arithmetic. If the beginnings of arithmetic appear self-evident, the rest of it, too, bears this character. Owing to this deductive character of arithmetic, and to its exemption from influence from without, this science appears to one person extremely attractive, while to another it appears extremely repulsive, according as each is constituted. Be that as it may, however, a finished and complete science of this character subserves no purpose in the comprehension of the world, or in the advancement of civilisation. Hence, an arithmetic which heaps up theorem on theorem with never a thought of how its results are to be turned to practical account in the acquisition of knowledge in other fields, resembles an inquisitive physician, who, taking up his abode in a desert, should arrive there at momentous results in bacteriology, but should bear them with him to his grave, without their ever redounding to the benefit of humanity. The value of

arithmetical knowledge lies entirely in its applications. But this constitutes no reason why many mathematicians, pursuing their purely deductive bent of mind, should not devote themselves exclusively to pure arithmetical developments and leave it to others at the proper time to turn to the material profit of the world the capital which they have garnered.

Geometry, on the other hand, must have recourse in a much higher degree than arithmetic to the outside world for its first notions and premises. The axioms of geometry are nothing but facts of experience perceived by our senses. The geometry which Bolyai, Lobatchewsky, Gauss, Riemann, and Helmholtz created and which is both independent of the eleventh axiom of Euclid and perfectly free from self-contradictions, has supplied an epistemological demonstration that geometry is a science that rests on the observation of nature, and therefore in the correct sense of the word, is a natural science.

Yet what a difference there is, for instance, between geometry and chemistry. Both derive their constructive materials from sense-perception. But whilst geometry is compelled to draw only its first results from observation and is then in a position to move forward deductively to other results without being under the necessity of making fresh observations, chemistry on the other hand is still compelled to make observations and to have recourse to nature.

It follows, therefore, that a given act of geometrical knowledge and a given act of chemical knowledge are with respect to the certainty of the truth they contain not qualitatively but only quantitatively different. In chemistry the probability of error is greater than in geometry, because more numerous and more difficult observations have to be made there than in geometry, where only the very first premises, which no man with sound senses could ever impugn, rest on observation.

The preceding reflexions deprive mathematical knowledge of that degree of certainty and incontestability which is commonly attributed to it when we say a thing is "mathematically certain." This certainty is lessened still more as we pass to the semi-mathematical sciences, where mechanics has the first claim to our at-

tention. All the notions of mechanics, and consequently of all the other departments of physics, are composed, by multiplication or division, of three fundamental notions—length, time, and mass. That is to say, to the notions of geometry resting on length and its powers, two other fundamental notions, time and mass, are added, which, joined to that of length, lead to the notions of force, work, horse-power, atmospheric pressure, etc. The knowledge of mechanics, thus, highly certain though it be, is rendered less certain than that of geometry and *a fortiori* than that of arithmetic. The uncertainty of knowledge continues to increase in branches which are still more remote from mathematics, owing to the increasing complexity of the observational material which must here be put to the test.

Still, although mathematical knowledge does not lead to absolutely certain results, it yet invests known results with incomparably greater trustworthiness than does the knowledge of the other sciences. But after all, it remains a useless accumulation of capital so long as it is not turned to practical account in other sciences, such as metaphysics, physics, chemistry, biology, political economy, etc. Hence also arises an obligation on the part of the other sciences, so to shape their problems and investigations that they can be made susceptible of mathematical treatment. Then will mathematics gladly perform her duty. The moment a science has advanced far enough to permit of the mathematical formulation of its problems, mathematics will not be slow to treat and to solve these problems. Mathematical knowledge, aristocratic as it may appear by the greater certainty of its results, will, so far as the advancement of human kind is concerned, never be more than a useless mass of self-evident truths, unless it constantly places itself in the service of the other sciences.

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